

Optimizing transport efficiency on scale-free networks through assortative or dissortative topology

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We find that transport on scale-free random networks depends strongly on degree-correlated network topologies whereas transport on Erdős-Rényi networks is insensitive to the degree correlation. An approach for the tuning of scale-free network transport efficiency through assortative or dissortative topology is proposed. We elucidate that the unique transport behavior for scale-free networks results from the heterogeneous distribution of degrees.

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Introduction. Understanding transport process on networks [1] is a central problem in many fields, [1,2] ranging from social networks to natural or technique networks. For example, epidemic spreading [3,4] and transportation [5] are typical phenomena of transport related to social networks, while the world-wide web, the internet, the biological networks and the new network-based materials are technical or natural networks [2–6]. Usually, we can classify transport on networks into two categories according to whether the network flow conservation is observed for each node of networks: the contamination process [3,4] and the network flow problem [5–10].

The network topology has a profound implication for network transport [1–10]. One of the important topological features in the network structure is the tendency of vertices with a given degree to be connected to other vertices with similar degree (assortativity) or dissimilar degree (dissortativity) [11]. Many real networks exhibit this degree correlation among their nodes. Such correlation plays an important role in transport process on networks. For the epidemic spreading, lack of an epidemic threshold has been verified in assortative networks [4]. However, in the network flow problem, the role of such topological vertex correlation is still unclear. Understanding the role of the degree correlation on the network flow process is important not only to biological networks [1,2] and traditional transportation networks [5], but also to the design of new network-based materials. [6] Thus, some interesting questions are: the degree correlation improve or deteriorate network transport? Can network transport efficiency be optimized through assortative or dissortative topology?

In this report, we first present the results of transport on scale-free random networks and Erdős-Rényi networks with different degree-correlated topologies. Then, a further comparison with two empirical networks is made.

Transport on networks. Let $G=(N,A)$ be a network defined by a set N of nodes and a set A of edges. Each edge (i,j) connected from node i to node j has an associated cost

function c_{ij} , which denotes the cost per unit flow on that edge. With the quadratic cost function, we can solve the maximum flow problem through Kirchhoff's equations [7,12]. This quadratic restriction on the cost function captures the essential properties of many important physical networks [5,7]. Then these equations are solved using the itpack method [13] parallelly. To characterize the transport capability of a whole network [14], we average the network conductance G_{st} between each pair of nodes as

$$\langle G \rangle = \frac{1}{N(N-1)} \sum_{s,t \in N; s \neq t} G_{st}, \quad (1)$$

where s and t run from 1 to N . Larger average conductance $\langle G \rangle$ signifies a better transport capability of networks. From a statistical perspective, a probability density function [8] (pdf) $\phi(G)$ can be defined through $\phi(G)dG$, which denotes the probability that two nodes have conductance between G and $G+dG$. A cumulative distribution function (cdf) $F(G)$ can be described by $F(G) = \int_G^\infty \phi(G')dG'$.

We use the uncorrelated configuration model [15] to generate the uncorrelated scale-free networks with the degree distribution $P(k) \sim k^{-\gamma}$, where k is the number of links attached to the node. In fact, scale-free networks generated from the uncorrelated configuration model belong to the scale-free random networks. Uncorrelated Erdős-Rényi networks are constructed with the standard random method [16]. We further employ the algorithm of reshuffling links [17,18] to transform the existing uncorrelated network to correlated networks. During the following calculation, we utilize the Newman factor [11] and the mean nearest neighbor function [11] $K_{nn}(k)$ to measure the degree-degree correlation property of networks.

Scale-free random networks. The scale-free networks with size $N=3000$ are constructed using the aforementioned method. To simplify the calculations, we set the value one to the weight of each edge in the networks, $v_s=1$ and $v_t=0$ for the source and sink nodes, respectively. The pdf $\phi(G)$ and cdf $F(G)$ vs G for scale-free networks with different degree correlations are illustrated in Fig. 1.

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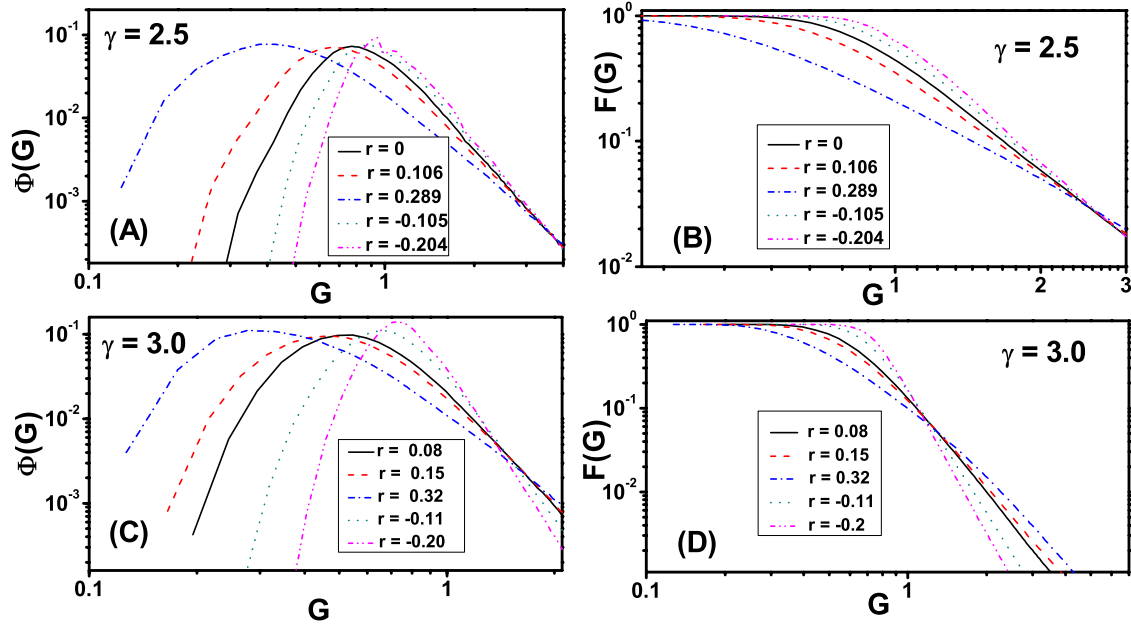


FIG. 1. (Color online) The pdf $\phi(G)$ and the cdf $F(G)$ vs G for different degree-correlated scale-free networks with $\gamma=2.5$ and $\gamma=3.0$. The values of Newman factors r are illustrated in each figure.

It can be seen from Figs. 1(A) and 1(C) that the peaks for the pdf $\phi(G)$ shift to higher G when the topological structures vary from assortativity to dissortativity. The change of average network conductance $\langle G \rangle$ against the degree correlation coefficient (Newman factor) r is plot in Fig. 2(A). It can be found that $\langle G \rangle$ is significantly reduced when the network topology is assortative. In contrast, the network conductance has been remarkably improved for the dissortative topology. To quantify such variation in average network conductance with the degree-mixing properties, we introduce the

metrics of network transport efficiency as

$$\eta = \frac{\langle G(r) \rangle - \langle G(r=0) \rangle}{\langle G(r=0) \rangle}, \quad (2)$$

where $\langle G(r) \rangle$ is average conductance for the network with the Newman factor r and $\langle G(r=0) \rangle$ is average network conductance for the uncorrelated networks. For $\gamma=2.5$, we can find that the network transport efficiency η decreases by 27% at $r=0.289$, in contrast with an increase of 21% at $r=-0.30$

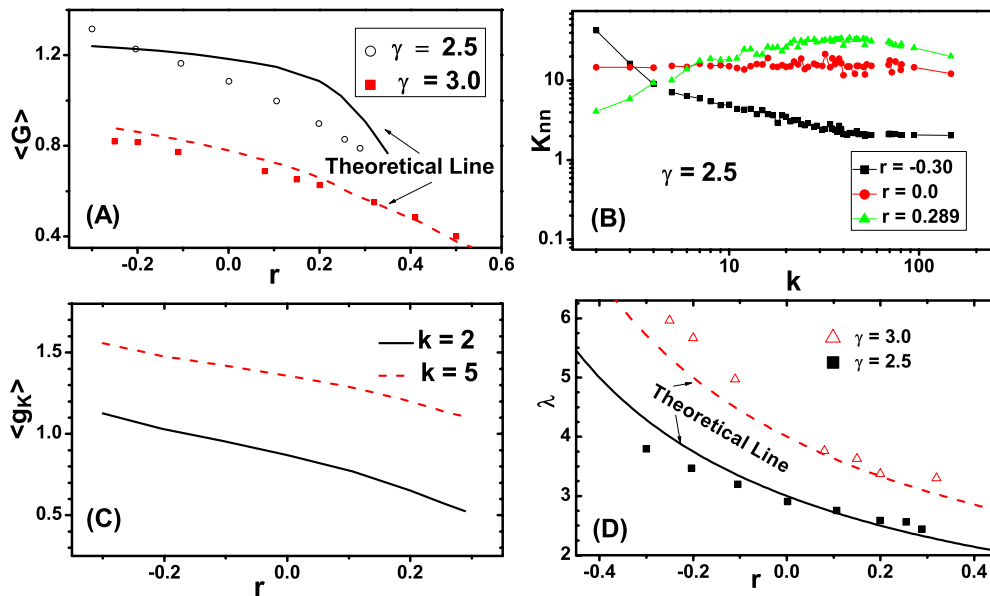


FIG. 2. (Color online) (A) Average network conductance $\langle G \rangle$ as a function of the Newman factor r . Solid line and dash lines are theoretical results calculated from Eq. (3). (B) The mean nearest neighbor function $K_{nn}(k)$ vs the degree k with different degree correlations. (C) Degree-averaged network conductance g_k changes with r . (D) The numerically fitted exponents λ for the power-law tail against r . Solid and dash lines are theoretical predictions.

as shown in Fig. 2(A). The scale-free network with $\gamma=3.0$ shows a similar behavior. Thus we can propose that the transport efficiency on scale-free networks be controlled by tuning the assortative/assortative network topology.

The reason why transport efficiency on scale-free networks depends strongly on the degree correlation lies in the fact that *the network conductance between low-degree nodes can be effectively changed through tuning the degree correlation topology*. To demonstrate this, in Fig. 2(C) we plot the change of the degree-averaged network conductance against the Newman factor r for the small-degree nodes. Here the degree-averaged network conductance $\langle g_k \rangle$ means that the network conductance is averaged over the pairs of nodes, where either or both nodes have the degree k . It can be seen that $\langle g_k \rangle$ for both $k=2$ and $k=5$ show an evident decrease for the assortative topology, in contrast with a significant increase for the dissortative topology.

The straightforward understanding for such a change of network conductance for the low-degree nodes with the degree correlation topology is as follows. For the dissortative topology, nodes with small degrees tend to be connected to other high-degree nodes such that more parallel branches are possible for the in/out-flows to/from the low-degree nodes. Therefore, the network conductance for the low-degree nodes under dissortativity is enhanced due to the presence of the neighboring high-degree nodes. In contrast, for the assortative topology, nodes with small degrees appear to be connected with the similarly low-degree nodes, which results in a reduced number of possible parallel flow branches. Thus, the network conductance for the low-degree nodes with the assortative topology decrease correspondingly. To verify this, the mean nearest neighbor functions $K_m(k)$ is plotted in Fig. 2(B) for the uncorrelated and correlated scale-free networks.

To document the above analysis, we further carry out a heuristical derivation of network conductance in the picture of transport backbone using a simplified branching process. [19] If we ignore the loops on networks, the conductance between node A and node B can be modeled [19] by a transport backbone with the average branching factor $\beta(k) = K_m(k) - 1$. In fact, the neglect of loops is reasonable when the networks considered are sparsely connected. So it is straightforward that the conductance g_a between node A and the infinite distance can be derived from the recursion relations [10,19] as $g_a = k(1 - 1/\beta)$. For scale-free networks, we only need to concentrate on the nodes with small degrees because most nodes are distributed in the range of low degrees. So we assume $\beta(k) \sim \alpha k^r - 1$, where α is a constant, depending on the degree-mixing property of networks and r is the Newman factor. It can be seen from Fig. 2(B) that this relation holds for small values of degrees. Thus, the conductance G_{ab} between node A and node B can be approximately written as $G_{ab} = g(k)/2$, provided that the values of k_a and k_b are small. The average conductance $\langle G \rangle$ on the network therefore can be written as

$$\langle G \rangle \approx \frac{1}{2} \int_{k_{min}}^{k_c} g(k) P(k) dk, \quad (3)$$

where k_c is the cutoff value for small degrees and $P(k) \sim k^{-\gamma}$. The calculated average network conductance by Eq.

(3) is plotted in Fig. 2(A). We can find that the calculated tendencies are qualitatively consistent with the numerical simulations. The discrepancy between numerical and theoretical results may come from loops and specific topology structures, which have been neglected in the branching process.

When we average the scale-free network conductance between each pair of nodes, the contributions from the low-degree nodes are dominant by virtue of large numbers of low-degree nodes. Therefore, network transport efficiency on scale-free networks changes correspondingly with the degree correlation. Similar reason accounts for the peak shift for the pdf $\phi(G)$ as shown in Fig. 1.

In addition, a power-law tail distribution of network conductance can be observed in both pdf $\phi(G)$ and cdf $F(G)$ as illustrated in Fig. 1. Such a power-law tail has been reported in Ref. [8] for the uncorrelated scale-free networks. Here we find the scaling exponent of the power-law tail is related to the degree correlation. In Fig. 2(D), we fit the cumulative power-law distribution as $F(G) \sim G^{-\lambda}$. The fitted power-law exponent λ clearly shows a tendency to decrease with the Newman factor r .

We can understand this variation in λ with the degree correlation through an analytical derivation. Owing to the fact that the distribution of conductance G_{ab} is characterized by the distribution of nodes with small degrees, the conditional probability [8] $Pr(k_a < k_b)$ with the constraint $k_a < k_b$ can be described as $Pr(k_a < k_b) dk_a \propto k_a^{-2\gamma+1} dk_a$. Here, we further make an explicit assumption for g_a associated with r as $g_a \sim k_a^{1+r}$, i.e., $G_{ab} \sim k_a^{1+r}$ for small-degree nodes. Such assumption agrees with the maximum-flow problem which states that the average maximum-flow between a pair of nodes is proportional to the smaller degree k . [20] We thus obtain distributions of the cdf $F(G) \sim G^{-(2\gamma-2)/(1+r)}$. It can be seen from Fig. 2(D) that the predictions of $\lambda_{theory} = (2\gamma - 2)/(1+r)$ shows the similar tendency as the fitted results.

Erdős-Rényi random networks. Network conductance for Erdős-Rényi networks with size of $N=3000$, and average degree $\bar{k}=8$ is plotted in Fig. 3(A). In contrast with scale-free networks, pdf $\phi(G)$ and average network conductance $\langle G \rangle$ for Erdős-Rényi networks show no significant dependence on degree correlation topologies. Although the tuning of degree correlation can induce some changes in network conductance for small-degrees nodes, but most nodes are distributed around the average degree \bar{k} due to the Poisson degree distribution. Therefore, the contributions from the low-degree nodes are trivial such that the tuning of degree correlation does not significantly alter the average network conductance.

Empirical networks. We calculate network conductance in two empirical networks: the Autonomous Systems (AS) level of the Internet and the U.S. Power Grid (PG) networks. The degree distributions of the two networks show a scale-free characteristics, with $\gamma=2.21 \pm 0.08$ for the AS network and $\gamma=3.7 \pm 0.2$ for the power grid networks. The Newman factors r are $r_{AS} = -0.2149$ and $r_{PG} = -0.03$, respectively. In Fig. 3(B), the fitted exponents λ of $F(G) \sim G^{-\lambda}$ are $\lambda_{AS} = 2.46 \pm 0.05$ and $\lambda_{PG} = 4.74 \pm 0.08$, respectively. We

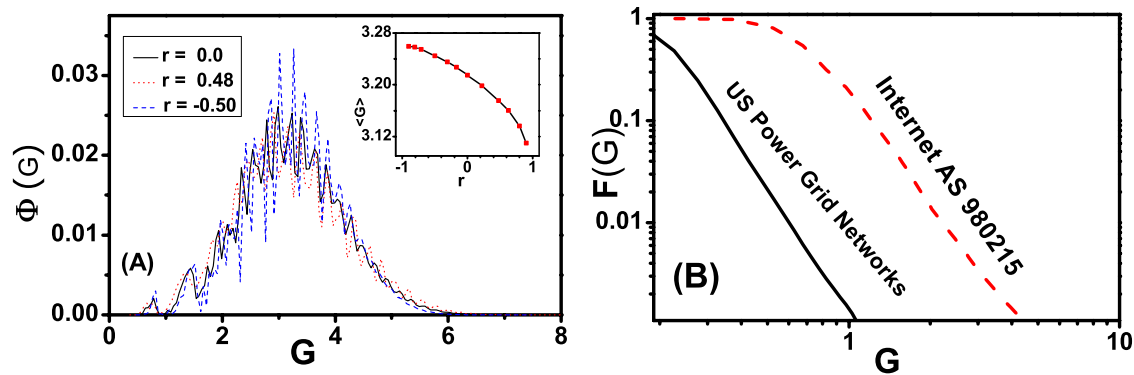


FIG. 3. (Color online) (A) The pdf $\phi(G)$ for Erdős-Rényi networks with different degree correlations. The inset shows the change of average network conductance as a function of Newman factor r . (B) The cdf $F(G)$ for two empirical networks: the internet structure [21] at the autonomous system level (AS980215) and the U.S. power grid networks [22].

theoretically calculate the scaling exponent λ as $\lambda_{AS}^{theory} = 3.08$ and $\lambda_{PG}^{theory} = 5.24$. The results for empirical networks are qualitatively consistent with that of the scale-free random network model. The large discrepancy between the real networks and the theoretical predictions may come from specific network topologies and loops that have been neglected in the theoretical formula.

In summary, we have found that transport on scale-free random networks vary significantly with the degree correlations. An approach for the tuning of transport efficiency on scale-free networks through dissortative or assortative topol-

ogy is proposed. We elucidate that the unique transport behavior for scale-free networks results from the heterogeneity of the degree distributions. We believe that our results provide some insights into the design of network transport.

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